

EXCEPTIONAL MINIMAL SURFACES IN SPHERES, PSEUDOHOLOMORPHIC CURVES IN S^6 AND POLAR SURFACES

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ABSTRACT. We deal with a class of minimal surfaces in spheres for which all Hopf differentials are holomorphic, the so called exceptional surfaces. We obtain a characterization of exceptional surfaces in any sphere in terms of a set of scalar invariants, the a -invariants, that satisfy certain Ricci-type conditions. We show that these surfaces are determined by the a -invariants up to a multiparameter family of isometric minimal deformations, where the number of the parameters is precisely the number of non-vanishing Hopf differentials. We give applications to superconformal surfaces and in particular to pseudoholomorphic curves in the nearly Kähler sphere S^6 . Moreover, we study superconformal surfaces in odd dimensional spheres that are isometric to their polar and show how they are related to pseudoholomorphic curves in S^6 .

1. INTRODUCTION

Minimal surfaces in the Euclidean space \mathbb{R}^3 are locally constructed via the Weierstrass representation. More generally, any minimal surface in \mathbb{R}^n is locally the real part of the integral of an isotropic curve in \mathbb{C}^n . The study of minimal surfaces in spheres was initiated by Calabi in his seminal paper [5] and follows a different completely route.

In this paper, we consider *exceptional* surfaces, a class of minimal surfaces in spheres for which certain invariants, the so called Hopf differentials, are holomorphic. There is an abundance of exceptional surfaces. This class of surfaces, includes the superminimal ones, which are the minimal surfaces with vanishing Hopf differentials. Minimal two-spheres are indeed superminimal [7]. Superconformal surfaces [4] are minimal surfaces that are characterized by the fact that the Hopf differentials vanish up to the last but one, are exceptional. Pseudoholomorphic curves in the nearly Kähler sphere S^6 are included in this class. Besides flat minimal surfaces (cf. [16, 17]), Lawson's surfaces, i.e., minimal surfaces that decompose as a direct sum of the associated minimal surfaces in S^3 are indeed exceptional (see [25]). These surfaces are related to Lawson's conjecture [19] which asserts that the only non-flat minimal surfaces in spheres that are locally isometric to minimal surfaces in S^3 are Lawson's surfaces.

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²We note that this terminology was used by Johnson [15], but his surfaces are superconformal (resp. superminimal) according as they lie in an odd-(resp. even-) dimensional sphere with substantial codimension.

The Hopf differentials are defined from the higher fundamental forms and the complex structure in the following way. Let $f : (M, ds^2) \rightarrow S^n$ be a minimal surface, i.e., an isometric minimal immersion of an oriented two-dimensional Riemannian manifold (M, ds^2) into S^n . The r -th normal space $N_p^r f$ of f has dimension ≤ 2 for any point $p \in M$. A point p is called generic (cf. [7, 1]) if $\dim N_p^r f = 2$ for any r , unless $r = m$ and $n = 2m + 1$, where $\dim N_p^m f = 1$. If f is substantial in S^n in the sense of [9], then the set of generic points is open and dense (cf. [21, p. 96]). Hereafter, we always assume that the minimal surfaces under consideration are substantial, unless otherwise stated. At generic points we can consider the r -th normal bundle $N^r f$ of f , with fibers $N_p^r f$. The $(r + 1)$ -th fundamental form of f is a symmetric $(r + 1)$ -linear tensor B_r from $T_p M$ into $N_p^r f$.

The complexified tangent bundle $TM \otimes \mathbb{C}$ is decomposed into the eigenspaces of the complex structure J , called $T'M$ and $T''M$, corresponding to the eigenvalues i and $-i$. The complex structure of M is given by the orientation and the induced metric. The $(r + 1)$ -th fundamental form B_r , which takes values in $N^r f$, can be complex linearly extended to $TM \otimes \mathbb{C}$ with values in the complexified vector bundle $N^r f \otimes \mathbb{C}$ and then decomposed into its (p, q) -components, $p + q = r + 1$, which are tensor products of p many 1-forms vanishing on $T''M$ and q many 1-forms vanishing on $T'M$. The minimality of f implies that the (p, q) -components of B_r vanish, unless $p = r + 1$ or $p = 0$, and consequently for a local complex coordinate z on M , we have the following decomposition

$$B_r = B_r^{(r+1,0)} dz^{r+1} + B_r^{(0,r+1)} d\bar{z}^{r+1}.$$

The Hopf differentials are by definition the differential forms

$$(1.1) \quad \Phi_r := \langle B_r^{(r+1,0)}, B_r^{(r+1,0)} \rangle dz^{2r+2},$$

of type $(2r + 2, 0)$, $r = 1, \dots, [(n - 1)/2]$, where $[(n - 1)/2]$ stands for the integer part of $(n - 1)/2$, and $\langle \cdot, \cdot \rangle$ denotes the extension of the usual Riemannian metric of S^n to a complex-valued complex bilinear form. These forms are defined at generic points and are independent of the choice of coordinates, while Φ_1 is globally well defined. It is a remarkable consequence of the structure equations that Φ_1 is always holomorphic (cf. [7, 8]).

The Hopf differentials play the role that Frenet curvatures play for curves, in the sense that two isometric minimal surfaces with the same Hopf differentials are congruent [24]. We note that Φ_r vanishes precisely at points where the r -th curvature ellipse is a circle [24, 25]. The r -th curvature ellipse [6] at a point p is the image of the unit circle on the tangent plane of M at p under the $(r + 1)$ -th fundamental form.

Using a null basis for each higher complexified normal bundle, we split the Hopf differentials into a product of two factors. The modulus of each factor defines scalar invariants which we call *a-invariants*. Their geometric meaning is that they determine the geometry of the higher curvature ellipses.

Our aim is to give a complete characterization of exceptional surfaces in spheres in terms of the *a*-invariants, and to characterize their induced metrics. In fact, we give an existence and uniqueness theorem for exceptional surfaces in terms of the *a*-invariants as in [11, 12]. We prove that for each exceptional surface, the *a*-invariants are of absolute value type functions in the sense of [11, 13] and satisfy certain conditions. Then we can reverse this process and show that each set of absolute value type functions that fulfill these conditions determines an exceptional surface

up to a multiparameter family of isometric minimal deformations. The number of the parameters is precisely the number of non-vanishing Hopf differentials.

Several applications are provided. At first, we give another short proof of the Lawson's conjecture for exceptional surfaces in odd-dimensional spheres [25]. Moreover, we give applications to pseudoholomorphic curves in the nearly Kähler sphere S^6 . We provide extrinsic and intrinsic characterizations for each type of such curves (see [3]) in terms of the a -invariants, among the class of superconformal surfaces.

It is well known [18] that the Gauss map of a minimal surface M in S^3 defines another minimal, the polar of M , which is conformal to M . On the other hand, the polar can also be defined for any minimal surface lying in an odd-dimensional sphere, just by selecting a unit section of the last normal bundle. It was proved by Miyaoka [20] that the polar of any superconformal surface is again a superconformal surface. Moreover, this construction is dual, in the sense that taking the polar a second time produces the original surface.

We investigate superconformal surfaces that are isometric to their polar, which we briefly call *self-dual surfaces*. In contrast to the case of S^3 where the Clifford torus is the only self-dual surface, there are lots of self-dual surfaces in high codimension. Flat superconformal surfaces in odd-dimensional surfaces are self-dual. Self-dual surfaces are related to the Ricci condition. It turns out that superconformal surfaces in S^{8k+7} that satisfy the Ricci condition are self-dual. In particular, Lawson's surfaces in S^{8k+7} that are superconformal are self-dual.

The case of self-dual surfaces in S^5 is quite interesting. We show that a superconformal surface in S^5 is self-dual if and only if it is congruent to a pseudoholomorphic curve of S^6 lying in a totally geodesic S^5 . More generally, we fully characterize all self-dual surfaces. It turns out that the property of being self-dual is intrinsic.

The paper is organized as follows: In section 2, we fix the notation and give some preliminaries. In section 3, we consider the splitting of the Hopf differentials and introduce the a -invariants. In section 4, we prove the main result of the paper, namely that the a -invariants determine all exceptional surfaces up to a multiparameter family. In section 5, we deal with pseudoholomorphic curves in the nearly Kähler sphere S^6 . Section 6 is devoted to the Ricci condition. In Section 7, we investigate self-dual surfaces. Finally some global formulas and topological restrictions are obtained.

2. PRELIMINARIES

Let $f : (M, ds^2) \rightarrow S^n$ be a minimal surface. Curves on M through a point $p \in M$ have their first derivatives on the tangent plane $T_p M$, but higher order derivatives will have components normal to f . The space $T_p^r f$ spanned by the derivatives of order up to r is called the r -th osculating space of f at p (cf. [7, 22]). Obviously, $T_p^r f$ is a subspace of $T_p^{r+1} f$. The r -th normal space of f at p , denoted by $N_p^r f$, is the orthogonal complement of $T_p^r f$ in $T_p^{r+1} f$, i.e., $T_p^{r+1} f = N_p^r f \oplus T_p^r f$. The $(r+1)$ -th fundamental form B_r is the $(r+1)$ -linear tensor from $T_p M$ into $N_p^r f$, defined by

$$B_r(X_1, \dots, X_{r+1}) = \pi_r(\bar{\nabla}_{\bar{X}_1} \dots \bar{\nabla}_{\bar{X}_r} \bar{X}_{r+1}),$$

where π_r is the orthogonal projection onto $N_p^r f$, $\bar{\nabla}$ is the Levi-Civita connection of S^n , and $\bar{X}_1, \dots, \bar{X}_{r+1}$ are local vector fields that extend X_1, \dots, X_{r+1} . It is well known that B_r is symmetric (cf. [22, p. 240]) and $N_p^r f$ is spanned by the image of B_r . Clearly, B_1 is nothing but the second fundamental form.

At generic points, we can consider the r -th normal bundle $N^r f$, with fibers $N_p^r f$. It is clear that $N^r f$ is a 2-vector bundle, except for the case when $n = 2m + 1$, where $N^m f$ is a 1-vector bundle.

We use the moving frame method and adopt the following convention on the range of indices (the symbol i is reserved for $\sqrt{-1}$), unless otherwise stated:

$$1 \leq j, k \leq 2, \quad 3 \leq \alpha, \beta \leq n, \quad 1 \leq A, B, C \leq n, \quad 1 \leq r, s, t \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Let $\{e_A\}$ be a local orthonormal frame field on S^n , and let $\{\omega_A\}$ be the coframe dual to $\{e_A\}$. The structure equations of S^n are

$$(2.1) \quad d\omega_A = \sum_B \omega_{AB} \wedge \omega_B,$$

$$(2.2) \quad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \omega_A \wedge \omega_B,$$

where the connection form ω_{AB} is given by $\omega_{AB}(X) = \langle \bar{\nabla}_X e_A, e_B \rangle$ and $\langle \cdot, \cdot \rangle$ is the Riemannian metric on S^n . We choose the frame such that, restricted to M , e_j is tangent and consequently e_α is normal to the surface. Then we have $\omega_\alpha = 0$. By (2.1) and Cartan's Lemma, we get

$$\omega_{j\alpha} = \sum_k h_{jk}^\alpha \omega_k, \quad h_{jk}^\alpha = h_{kj}^\alpha.$$

The assumption that f is minimal is equivalent to $h_{11}^\alpha + h_{22}^\alpha = 0$. Restricting equations (2.1) and (2.2) to M , we get the Cartan structure equations of f .

Hereafter we set $m := \lfloor (n-1)/2 \rfloor$, and choose the normal frame e_α such that (e_{2r+1}, e_{2r+2}) is a frame field of $N^r f$ for any $r \leq m$. When $n = 2m + 1$, e_{2m+1} spans the fibers of $N^m f$. Then it is easy to see that (cf. [22, Lemma 69])

$$(2.3) \quad \omega_{2r-1, \alpha} = \omega_{2r, \alpha} = 0 \quad \text{if} \quad \alpha > 2r + 2 \quad \text{or} \quad \alpha < 2r - 3.$$

The components of the higher fundamental forms are given by

$$h_1^\alpha := \langle B_r(e_1, \dots, e_1), e_\alpha \rangle, \quad h_2^\alpha := \langle B_r(e_1, \dots, e_1, e_2), e_\alpha \rangle,$$

where $\alpha = 2r + 1$ or $2r + 2$. We use complex vectors, and we put

$$H_\alpha = h_1^\alpha + ih_2^\alpha, \quad E = e_1 - ie_2 \quad \text{and} \quad \varphi = \omega_1 + i\omega_2.$$

Then we have (cf. [7, p. 30]):

$$(2.4) \quad H_{2r+1}\omega_{2r+1, \alpha} + H_{2r+2}\omega_{2r+2, \alpha} = H_\alpha \bar{\varphi} \quad \text{for} \quad \alpha = 2r + 3, 2r + 4,$$

$0 \leq r \leq m - 1$, where $H_1 = 1, H_2 = i$, and when $n = 2m + 1$

$$H_{2m-1}\omega_{2m-1, 2m+1} + H_{2m}\omega_{2m, 2m+1} = H_{2m+1}\bar{\varphi}.$$

The induced metric is $ds^2 = \varphi \bar{\varphi}$. From (2.1) we find

$$(2.5) \quad d\varphi = -i\omega_{12} \wedge \varphi.$$

The Gaussian curvature K is given by

$$d\omega_{12} = -\frac{i}{2}K\varphi \wedge \bar{\varphi}.$$

We choose a local complex coordinate $z = x + iy$ such that $\varphi = Fdz$. Then

$$B_r = B_r^{(r+1, 0)} dz^{r+1} + B_r^{(0, r+1)} d\bar{z}^{r+1},$$

where

$$B_r^{(r+1,0)} = B_r(\partial, \dots, \partial) \quad \text{and} \quad \partial = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

Therefore, from the definition (1.1) of Hopf differentials, we easily get

$$\Phi_r = \frac{1}{4} \left(\overline{H}_{2r+1}^2 + \overline{H}_{2r+2}^2 \right) \varphi^{2r+2}.$$

The r -th curvature ellipse, for each point p in M , is the subset of $N_p^r f$ given by

$$\mathcal{E}_r(p) = \{ B_r(X, \dots, X) \in N_p^r f : X \in T_p M, |X| = 1 \}.$$

It is known (cf. [6]) that $\mathcal{E}_r(p)$ is indeed an ellipse (possibly degenerated). The r -th normal curvature K_r^\perp is defined by

$$(2.6) \quad K_r^\perp = i \left(H_{2r+1} \overline{H}_{2r+2} - \overline{H}_{2r+1} H_{2r+2} \right).$$

It is not hard to verify that

$$|K_r^\perp| = \frac{2}{\pi} \text{Area}(\mathcal{E}_r).$$

We note that the sign of K_r^\perp depends on the orientation of the bundle $N^r f$. It is obvious that $K_r^\perp(p) = 0$ if and only if $\dim N_p^r f \leq 1$. Let $\kappa_r \geq \mu_r \geq 0$ be the length of the semi-axes of \mathcal{E}_r . Then

$$(2.7) \quad |K_r^\perp| = 2\kappa_r \mu_r.$$

For the sake of convenience, we also set $K_0^\perp = 2$. The length of B_r is given by

$$(2.8) \quad \|B_r\|^2 = 2^r (|H_{2r+1}|^2 + |H_{2r+2}|^2),$$

or equivalently (cf. [1])

$$(2.9) \quad \|B_r\|^2 = 2^r (\kappa_r^2 + \mu_r^2).$$

Moreover, using (2.7) and (2.8), we see that

$$(2.10) \quad \left| \langle B_r^{(r+1,0)}, B_r^{(r+1,0)} \rangle \right|^2 = \frac{F^{2r+2}}{2^{2r+4}} \left(\|B_r\|^4 - 4^r (K_r^\perp)^2 \right).$$

In particular, the zeros of Φ_r are precisely the points where \mathcal{E}_r is a circle.

On account of (2.6), from (2.4) and its conjugation, we get

$$(2.11) \quad \omega_{2r+1,2r+3} = \frac{i}{K_r^\perp} \left(H_{2r+3} \overline{H}_{2r+2} \overline{\varphi} - H_{2r+2} \overline{H}_{2r+3} \varphi \right),$$

$$(2.12) \quad \omega_{2r+2,2r+3} = \frac{i}{K_r^\perp} \left(H_{2r+1} \overline{H}_{2r+3} \varphi - H_{2r+3} \overline{H}_{2r+1} \overline{\varphi} \right),$$

$$(2.13) \quad \omega_{2r+1,2r+4} = \frac{i}{K_r^\perp} \left(H_{2r+4} \overline{H}_{2r+2} \overline{\varphi} - H_{2r+2} \overline{H}_{2r+4} \varphi \right),$$

$$(2.14) \quad \omega_{2r+2,2r+4} = \frac{i}{K_r^\perp} \left(H_{2r+1} \overline{H}_{2r+4} \varphi - H_{2r+4} \overline{H}_{2r+1} \overline{\varphi} \right),$$

at points where $K_r^\perp \neq 0, r \geq 0$.

Taking the exterior derivative of (2.4), conjugating and using (2.2)-(2.5), we obtain

$$(2.15) \quad \begin{aligned} -i\overline{H}_\alpha\omega_{12}\wedge\varphi + d\overline{H}_\alpha\wedge\varphi &= d\overline{H}_{\alpha-2}\wedge\omega_{\alpha-2,\alpha} + d\overline{H}_{\alpha-1}\wedge\omega_{\alpha-1,\alpha} \\ &+ \omega_{\alpha-2,\alpha-1}\wedge(\overline{H}_{\alpha-2}\omega_{\alpha-1,\alpha} - \overline{H}_{\alpha-1}\omega_{\alpha-2,\alpha}) \\ &+ \overline{H}_{\alpha+1}\varphi\wedge\omega_{\alpha+1,\alpha} \end{aligned}$$

for $\alpha = 2r + 3, r \geq 0$, and

$$(2.16) \quad \begin{aligned} -i\overline{H}_\alpha\omega_{12}\wedge\varphi + d\overline{H}_\alpha\wedge\varphi &= d\overline{H}_{\alpha-3}\wedge\omega_{\alpha-3,\alpha} + d\overline{H}_{\alpha-2}\wedge\omega_{\alpha-2,\alpha} \\ &+ \omega_{\alpha-3,\alpha-2}\wedge(\overline{H}_{\alpha-3}\omega_{\alpha-2,\alpha} - \overline{H}_{\alpha-2}\omega_{\alpha-3,\alpha}) \\ &+ \overline{H}_{\alpha-1}\varphi\wedge\omega_{\alpha-1,\alpha} \end{aligned}$$

for $\alpha = 2r + 4, r \geq 0$.

Each 2-vector bundle $N^r f$ inherits a Riemannian connection from that of the normal bundle of f . The intrinsic curvature K_r^* of each plane bundle $N^r f$ is defined, up to orientation, by

$$d\omega_{2r+1,2r+2} = -K_r^*\omega_1\wedge\omega_2.$$

The following proposition is due to Asperti [1].

Proposition 1. *The intrinsic curvature K_r^* of each plane bundle $N^r f$ of a minimal surface $f : (M, ds^2) \rightarrow S^n$ is given by*

$$K_1^* = K_1^\perp - \frac{\|B_2\|^2}{2K_1^\perp}; \quad K_r^* = \frac{K_r^\perp\|B_{r-1}\|^2}{2^{r-2}(K_{r-1}^\perp)^2} - \frac{\|B_{r+1}\|^2}{2^r K_r^\perp}, \quad 2 \leq r \leq m-1.$$

We use the above mentioned notation throughout the paper.

3. THE SPLITTING OF HOPF DIFFERENTIAL AND THE a -INVARIANTS

Using the null basis $\eta_r = e_{2r+1} + ie_{2r+2}, \overline{\eta}_r = e_{2r+1} - ie_{2r+2}$ of the complexified bundle $N^r f \otimes \mathbb{C}$, we have

$$\langle B_r^{(r+1,0)}, B_r^{(r+1,0)} \rangle = \langle B_r^{(r+1,0)}, \eta_r \rangle \langle B_r^{(r+1,0)}, \overline{\eta}_r \rangle.$$

Therefore, from the definition (1.1) of Hopf differentials, we easily get

$$4\Phi_r = \left(\overline{H}_{2r+1}^2 + \overline{H}_{2r+2}^2 \right) \varphi^{2r+2} = k_r^+ k_r^- \varphi^{2r+2},$$

where

$$k_r^\pm := \overline{H}_{2r+1} \pm i\overline{H}_{2r+2}.$$

Then we introduce the a -invariants as the scalar invariants

$$a_r^\pm := |k_r^\pm|.$$

The functions a_r^\pm are globally well defined. Their geometric meaning is that they both determine the geometry of the r -th curvature ellipse. Indeed, it follows from (2.6)-(2.9) that

$$a_r^\pm = 2^{-r} \|B_r\|^2 \pm K_r^\perp,$$

or equivalently

$$a_r^\pm = \kappa_r \pm \mu_r \quad \text{or} \quad a_r^\pm = \kappa_r \mp \mu_r,$$

depending on the orientation of $N^r f$. When $\Phi_r = 0$ for some r , then one of a_r^\pm vanishes. When this occurs, we make the convention that we choose the orientation so that a_r^- vanishes.

It will be convenient to set $a_0^+ = 2$ and $a_0^- = 0$. We now recall the definition of exceptional surfaces.

Definition 1. *A minimal surface $f : (M, ds^2) \rightarrow S^n$ is said to be exceptional if and only if all its Hopf differentials are holomorphic.*

The following characterization for exceptional surfaces in terms of the higher curvature ellipses was given in [25].

Theorem 1. *A minimal surface is exceptional if and only if its higher curvature ellipses have constant eccentricity up to the last but one.*

The following auxiliary lemma is needed for the proof of the main results.

Lemma 1. *For any exceptional surface $f : (M, ds^2) \rightarrow S^n$ and for any $1 \leq r \leq m$, where $m = [(n-1)/2]$, the following holds:*

$$dk_r^\pm - i(s+1)k_r^\pm \omega_{12} \pm ik_r^\pm \omega_{2r+1, 2r+2} \equiv 0 \pmod{\varphi}.$$

Proof. Using (2.15), (2.16) and arguing as in the proof of Proposition 4 in [25], we obtain

$$d\overline{H}_{2r+1} - i(r+1)\overline{H}_{2r+1}\omega_{12} - \overline{H}_{2r+2}\omega_{2r+1, 2r+2} \equiv 0 \pmod{\overline{\varphi}}$$

and

$$d\overline{H}_{2r+2} - i(r+1)\overline{H}_{2r+2}\omega_{12} + \overline{H}_{2r+1}\omega_{2r+1, 2r+2} \equiv 0 \pmod{\overline{\varphi}}.$$

Then the lemma follows directly. \square

4. A CHARACTERIZATION OF EXCEPTIONAL SURFACES

In this section we give our main results, according to which all exceptional surfaces are determined up to a multiparameter family of isometric minimal deformations by the a -invariants, provided that they satisfy certain restrictions.

For the proof of the results, we use the notion of absolute value type functions introduced in [11, 13]. A smooth complex valued function p defined on a connected oriented surface (M, ds^2) is called of *holomorphic type* if locally $p = p_0 p_1$, where p_0 is holomorphic and p_1 is smooth without zeros. A function $a : M \rightarrow [0, +\infty)$ on M is called of *absolute value type (AVT)* if there is a function p of holomorphic type on M such that $a = |p|$. The zero set of such a function is either isolated or the whole of M , and outside its zeros the function is smooth. We need the following two lemmas that were proved in [11, 13].

Lemma 2. *Let p be a smooth complex valued function defined on M , $p \neq 0$, and ω a real valued 1-form on M . Let $\psi := pdz$ for some conformal coordinate z . Then the equality*

$$d\psi = i\omega \wedge \psi$$

is valid if and only if p is of holomorphic type and

$$\omega = 2\operatorname{Im}(\overline{\partial}(\log p)d\overline{z}).$$

Moreover, then

$$d\omega = -\frac{1}{2i}\Delta \log |p|\overline{\varphi} \wedge \varphi.$$

It is worth mentioning that $\overline{\partial}(\log p)$ and $\Delta \log |p|$ are well defined even at the zeros of p , if p is of holomorphic type. Here Δ denotes the Laplace-Beltrami operator of (M, ds^2) .

Lemma 3. *Let a be an AVT function on an open, simply connected subset U of \mathbb{C} such that $\Delta^0 \log a = 0$, where Δ^0 is the Euclidean Laplacian. Then there exists a holomorphic function h on U with $a = |h|$.*

The function h in Lemma 3 is determined up to a factor $e^{i\theta}$.

For any minimal surface we set

$$\rho_r := 2^r \frac{|K_r^\perp|}{\|B_r\|^2} \quad \text{and} \quad \rho_0 := 1.$$

Obviously $\rho_r = 1$ precisely at points where Φ_r vanishes.

We may now state the main results.

Theorem 2. *Let $f : (M, ds^2) \rightarrow S^n$ be an exceptional surface with Gaussian curvature K and $m = \lfloor (n-1)/2 \rfloor$. Then the functions ρ_r are constant for any $1 \leq r \leq m-1$, the a -invariants are AVT and satisfy the following:*

$$(4.1) \quad a_r^- = \sigma_r a_r^+, 0 \leq r \leq m-1, \quad \sigma_r := \sqrt{\frac{1-\rho_r}{1+\rho_r}}, \quad a_1^+ = \sqrt{(1+\rho_1)(1-K)},$$

$$(4.2) \quad \Delta \log a_r^\pm = (r+1)K \mp \left(\frac{\rho_r b_r^2}{\rho_{r-1}^2 b_{r-1}^2} - \frac{b_{r+1}^2}{\rho_r b_r^2} \right), \quad 0 \leq r \leq m-1,$$

where

$$b_r := \sqrt{2^r/(1+\rho_r)} a_r^+, \quad 0 \leq r \leq m-1 \quad \text{and} \quad b_m := \|B_m\|.$$

Moreover

$$(4.3) \quad \Delta \log a_m^\pm = (m+1)K \mp \frac{2^m K_m^\perp}{\rho_{m-1}^2 b_{m-1}^2}.$$

Theorem 2 shows that a_r^\pm are intrinsic for any $1 \leq r \leq m-1$. This implies that $\|B_r\|$ and $|K_r^\perp|$ are intrinsic for any $1 \leq r \leq m-1$. In the case where f lies in $S^{2m+1} \subset S^{2m+2}$, we have $K_m^\perp = 0$ and $a_{m-1}^+ = a_{m-1}^-$ and so Theorem 2 shows that all a -invariants are intrinsic. Furthermore, the metric $(a_m^+)^{\frac{2}{m+1}} ds^2$ is flat.

It is worth noticing that any exceptional surface with non-vanishing first Hopf differential satisfies the Ricci condition, namely the metric $d\tilde{s}^2 = \sqrt{1-K} ds^2$ is flat away from points where $K = 1$ or equivalently $\Delta \log(1-K) = 4K$. This follows immediately from (4.1) and (4.2) for $r = 1$.

The converse of Theorem 2 is also true and can be stated in the following way.

Theorem 3. *Let (M, ds^2) be a simply connected two-dimensional Riemannian manifold with Gaussian curvature $K \not\equiv 1$. Let $0 < \rho_r \leq 1, 0 \leq r \leq m-1$, be constant numbers with $\rho_0 = 1$. Assume that there exist AVT functions $a_r^\pm, 1 \leq r \leq m-1$, and a non-negative function b_m so that*

$$(4.4) \quad a_r^- = \sigma_r a_r^+, 0 \leq r \leq m-1, \quad \sigma_r := \sqrt{\frac{1-\rho_r}{1+\rho_r}}, \quad a_1^+ = \sqrt{(1+\rho_1)(1-K)},$$

$$(4.5) \quad \Delta \log a_r^\pm = (r+1)K \mp \left(\frac{\rho_r b_r^2}{\rho_{r-1}^2 b_{r-1}^2} - \frac{b_{r+1}^2}{\rho_r b_r^2} \right), \quad 0 \leq r \leq m-1,$$

where

$$b_r := \sqrt{2^r/(1+\rho_r)} a_r^+, \quad 0 \leq r \leq m-1.$$

Let $K^\perp : M \rightarrow \mathbb{R}$ be a smooth function satisfying the inequality $|K^\perp| \leq 2^{-m}b_m^2$. If the functions

$$a_m^\pm := \sqrt{2^{-m}b_m^2 \pm K^\perp}$$

are AVT and satisfy

$$(4.6) \quad \Delta \log a_m^\pm = (m+1)K \mp \frac{2^m K^\perp}{\rho_{m-1}^2 b_{m-1}^2},$$

then for any $\theta_1, \dots, \theta_t \in \mathbb{R}$, where t is the number of ρ_r that are not equal to 1, increased by 1 if $a_m^+ a_m^- \neq 0$, there exists a minimal surface $f_{\theta_1, \dots, \theta_t} : (M, ds^2) \rightarrow S^n$, with $m = \lfloor (n-1)/2 \rfloor$ whose a -invariants are precisely the AVT functions a_r^\pm , $1 \leq r \leq m$. Furthermore, f is exceptional and any other minimal immersion of (M, ds^2) into S^n having the same a -invariants is congruent to some $f_{\theta_1, \dots, \theta_t}$ for appropriate $\theta_1, \dots, \theta_t$.

Superconformal surfaces have vanishing Hopf differentials up to the last but one. This means that $a_r^- = 0$ for any $0 \leq r \leq m-1$, and so the following corollary follows immediately from Theorems 2 and 3.

Corollary 1. *Let $f : (M, ds^2) \rightarrow S^n$ be a superconformal surface with Gaussian curvature K and set $m = \lfloor (n-1)/2 \rfloor$. Then $a_r^- = 0$, $1 \leq r \leq m-1$ and the functions a_r^+ , $1 \leq r \leq m-1$, are AVT and satisfy the following:*

$$(4.7) \quad \Delta \log a_r^+ = (r+1)K - \frac{b_r^2}{b_{r-1}^2} + \frac{b_{r+1}^2}{b_r^2}, \quad 0 \leq r \leq m-1,$$

where

$$a_1^+ = \sqrt{2(1-K)}, \quad b_r := 2^{\frac{r-1}{2}} a_r^+, \quad 0 \leq r \leq m-1 \quad \text{and} \quad b_m := \|B_m\|.$$

Moreover

$$(4.8) \quad \Delta \log a_m^\pm = (m+1)K \mp \frac{2^m K^\perp}{b_{m-1}^2}.$$

Conversely, let (M, ds^2) be a simply connected two-dimensional Riemannian manifold with Gaussian curvature $K \leq 1$. Assume that there exist AVT functions a_r^\pm , $1 \leq r \leq m-1$, and a non-negative function b_m that fulfill (4.7). If for a given smooth function $K^\perp : M \rightarrow \mathbb{R}$ satisfying the inequality $|K^\perp| \leq 2^{-m}b_m^2$, the functions

$$a_m^\pm := \sqrt{2^{-m}b_m^2 \pm K^\perp}$$

are AVT and satisfy (4.8), then for any $\theta \in \mathbb{R}$ there exists a minimal surface $f_\theta : (M, ds^2) \rightarrow S^n$, with $m = \lfloor (n-1)/2 \rfloor$, whose a -invariants are precisely the AVT functions a_r^\pm , $1 \leq r \leq m$, with $a_r^- = 0$ for any $0 \leq r \leq m-1$. Furthermore, f is superconformal and any other minimal immersion of (M, ds^2) into S^n having the same a -invariants is congruent to some f_θ . Furthermore, if $a_m^+ a_m^- = 0$, then n is even and f is rigid.

Miyaoka [20] determined all superconformal surfaces lying in spheres of odd dimension in terms of solutions of the corresponding affine Toda equations. The above corollary gives another characterization of superconformal surfaces in any sphere.

The class of superminimal surfaces has been investigated by various authors (cf. [2, 5, 7, 15]). As a result, superminimal surfaces are rigid, lie in even dimensional

spheres. Actually these results follow from Corollary 1. Furthermore, Theorems 2 and 3 extend earlier results due to Eschenburg, Tribuzy and Guadalupe [23, 12].

Proof of Theorem 2. The fact that $\rho_r, 1 \leq r \leq m-1$, are constant follows immediately from Theorem 1. Moreover, (4.1) is a consequence of the definition of a -invariants. We choose the frame in the normal bundle as in Section 2 and we put

$$\psi_r^\pm := k_r^\pm \varphi, \quad \omega_r^\pm := r\omega_{12} \mp \omega_{2r+1, 2r+2}, \quad 1 \leq r \leq m.$$

Assume that $\varphi = \mu dz$ for a local conformal coordinate z and set $\lambda = |\mu|$. Appealing to Lemma 1, we get

$$(4.9) \quad d\psi_r^\pm = i\omega_r^\pm \wedge \psi_r^\pm, \quad 1 \leq r \leq m.$$

Then Lemma 2 implies that the functions $a_r^\pm, 1 \leq r \leq m$, are AVT. Moreover,

$$d\omega_r^\pm = -\frac{1}{2i} \Delta \log |p_r^\pm| \bar{\varphi} \wedge \varphi,$$

where $\psi_r^\pm = p_r^\pm dz = k_r^\pm \mu dz$, or equivalently

$$rd\omega_{12} \mp d\omega_{2r+1, 2r+2} = -\frac{1}{2i} \Delta \log(a_r^\pm \lambda) \bar{\varphi} \wedge \varphi.$$

Now bearing in mind $d\omega_{12} = -\frac{i}{2} K \varphi \wedge \bar{\varphi}$, and the definition of the intrinsic curvature K_r^* of the bundle $N^r f$, from the above we get

$$\Delta \log(a_r^\pm \lambda) = rK \mp K_r^*.$$

Since $\Delta \log \lambda = -K$, we have

$$\Delta \log a_r^\pm = (r+1)K \mp K_r^*, \quad 1 \leq r \leq m.$$

Thus (4.2) and (4.3) follow from this, Proposition 1 and the equations

$$(a_r^\pm)^2 = 2^{-r}(1 + \rho_r) \|B_r\|^2 = \frac{1 + \rho_r}{\rho_r} K_r^\pm.$$

□

Proof of Theorem 3. (i) Existence. Let $U \subset M$ be an open, simply connected subset and $z = x + iy$ a conformal coordinate on U . Choose an orthonormal frame e_1, e_2 so that $\varphi = \mu dz$ and set $\lambda = |\mu|$.

Since by assumption the functions $a_r^\pm, 1 \leq r \leq m-1$, are AVT, there exist functions $k_r^\pm : U \rightarrow \mathbb{C}$ of holomorphic type such that $a_r^\pm = |k_r^\pm|$. In particular, we set $k_0^+ = 2$ and $k_0^- = 0$. We put

$$\psi_r^\pm := k_r^\pm \varphi = p_r^\pm dz, \quad 1 \leq r \leq m.$$

Appealing to Lemma 2, we get

$$(4.10) \quad d\psi_r^\pm = i\omega_r^\pm \wedge \psi_r^\pm, \quad 1 \leq r \leq m,$$

where ω_r^\pm is the real 1-form

$$(4.11) \quad \omega_r^\pm = 2\text{Im}(\bar{\partial}(\log p_r^\pm) d\bar{z}).$$

Moreover,

$$(4.12) \quad d\omega_r^\pm = -\frac{1}{2i} \Delta \log |p_r^\pm| \bar{\varphi} \wedge \varphi.$$

For any $\theta_r \in \mathbb{R}$, $0 \leq r \leq m-2$, we define the following real 1-forms

$$\begin{aligned}\omega_{2r+1,2r+3} &:= \frac{(1+\rho_r)(1+\sigma_{r+1})(1-\sigma_r)}{2\rho_r} \operatorname{Re}\left(\frac{e^{i\theta_{r+1}}k_{r+1}^+}{e^{i\theta_r}k_r^+}\varphi\right), \\ \omega_{2r+2,2r+3} &:= -\frac{(1+\rho_r)(1+\sigma_{r+1})(1+\sigma_r)}{2\rho_r} \operatorname{Im}\left(\frac{e^{i\theta_{r+1}}k_{r+1}^+}{e^{i\theta_r}k_r^+}\varphi\right), \\ \omega_{2r+1,2r+4} &:= \frac{(1+\rho_r)(1-\sigma_{r+1})(1-\sigma_r)}{2\rho_r} \operatorname{Im}\left(\frac{e^{i\theta_{r+1}}k_{r+1}^+}{e^{i\theta_r}k_r^+}\varphi\right), \\ \omega_{2r+2,2r+4} &:= \frac{(1+\rho_r)(1-\sigma_{r+1})(1+\sigma_r)}{2\rho_r} \operatorname{Re}\left(\frac{e^{i\theta_{r+1}}k_{r+1}^+}{e^{i\theta_r}k_r^+}\varphi\right).\end{aligned}$$

For any $1 \leq r \leq m-1$, we define

$$\omega_{2r+1,2r+2} := \begin{cases} r\omega_{12} - \omega_r^+ & \text{if } \rho_r = 1 \\ \frac{1}{2}(\omega_r^+ - \omega_r^-) & \text{if } \rho_r < 1. \end{cases}$$

Furthermore, for any $\theta_m \in \mathbb{R}$, we define

$$\begin{aligned}\omega_{2m-1,2m+1} &:= \frac{(1+\rho_{m-1})(1-\sigma_{m-1})}{2\rho_{m-1}} \operatorname{Re}\left(\frac{e^{i\theta_m}(k_m^+ + k_m^-)}{e^{i\theta_{m-1}}k_{m-1}^+}\varphi\right), \\ \omega_{2m,2m+1} &:= -\frac{(1+\rho_{m-1})(1+\sigma_{m-1})}{2\rho_{m-1}} \operatorname{Im}\left(\frac{e^{i\theta_m}(k_m^+ + k_m^-)}{e^{i\theta_{m-1}}k_{m-1}^+}\varphi\right), \\ \omega_{2m-1,2m+2} &:= \frac{(1+\rho_{m-1})(1-\sigma_{m-1})}{2\rho_{m-1}} \operatorname{Im}\left(\frac{e^{i\theta_m}(k_m^+ - k_m^-)}{e^{i\theta_{m-1}}k_{m-1}^+}\varphi\right), \\ \omega_{2m,2m+2} &:= \frac{(1+\rho_{m-1})(1+\sigma_{m-1})}{2\rho_{m-1}} \operatorname{Re}\left(\frac{e^{i\theta_m}(k_m^+ - k_m^-)}{e^{i\theta_{m-1}}k_{m-1}^+}\varphi\right), \\ \omega_{2m+1,2m+2} &:= \frac{1}{2}(\omega_m^+ - \omega_m^-).\end{aligned}$$

In all other cases, we define $\omega_{AB} = 0$.

Our aim is to prove that the forms ω_j and ω_{AB} satisfy the structure equations (2.1) and (2.2). Actually, we will only confirm that

$$(4.13) \quad d\omega_{2r+1,2r+2} = \sum_C \omega_{2r+1,C} \wedge \omega_{C,2r+2}, \quad 1 \leq r \leq m$$

and

$$(4.14) \quad d\omega_{2r+1,2r+3} = \sum_C \omega_{2r+1,C} \wedge \omega_{C,2r+3}, \quad 1 \leq r \leq m.$$

The proof of the rest structure equations follows in the same manner.

From (4.12), our assumption (4.5) and $\Delta \log \lambda = -K$, we have

$$2id\omega_r^\pm = \left\{ -rK \pm \left(\frac{\rho_r b_r^2}{\rho_{r-1}^2 b_{r-1}^2} - \frac{b_{r+1}^2}{\rho_r b_r^2} \right) \right\} \bar{\varphi} \wedge \varphi, \quad 1 \leq r \leq m-1.$$

If $\rho_r = 1$, then $\omega_{2r+1,2r+2} = r\omega_{12} - \omega_r^+$ and on account of $d\omega_{12} = -\frac{i}{2}K\varphi \wedge \bar{\varphi}$ we get

$$(4.15) \quad d\omega_{2r+1,2r+2} = -\frac{1}{2i} \left(\frac{\rho_r b_r^2}{\rho_{r-1}^2 b_{r-1}^2} - \frac{b_{r+1}^2}{\rho_r b_r^2} \right) \bar{\varphi} \wedge \varphi, \quad 1 \leq r \leq m-1.$$

If $\rho_r < 1$, then $\omega_{2r+1,2r+2} = \frac{1}{2}(\omega_r^+ - \omega_r^-)$. From (4.12) and (4.4), we obtain

$$d\omega_{2r+1,2r+2} = 0, \quad 1 \leq r \leq m-1.$$

Moreover, from (4.5) we get

$$\frac{\rho_r b_r^2}{\rho_{r-1}^2 b_{r-1}^2} - \frac{b_{r+1}^2}{\rho_r b_r^2} = 0.$$

This shows that (4.15) remains true in either case. On the other hand, bearing in mind the definition of the 1-forms ω_{AB} , we can see that

$$\sum_C \omega_{2r+1,C} \wedge \omega_{C,2r+2} = -\frac{i}{2} \left(\frac{b_{r+1}^2}{\rho_r b_r^2} - \frac{\rho_r b_r^2}{\rho_{r-1}^2 b_{r-1}^2} \right) \bar{\varphi} \wedge \varphi, \quad 1 \leq r \leq m-1.$$

Hence (4.13) holds true for any $1 \leq r \leq m-1$.

Using (4.12), $\omega_{2m+1,2m+2} = \frac{1}{2}(\omega_m^+ - \omega_m^-)$, and the assumption (4.5), we find

$$d\omega_{2m+1,2m+2} = -\frac{1}{2i} \frac{2^m K^\perp}{\rho_{m-1}^2 b_{m-1}^2} \bar{\varphi} \wedge \varphi.$$

On the other hand, bearing in mind the definition of the 1-forms ω_{AB} , by direct calculations we conclude that

$$\sum_C \omega_{2m+1,C} \wedge \omega_{C,2m+2} = \frac{i}{2} \frac{2^m K^\perp}{\rho_{m-1}^2 b_{m-1}^2} \bar{\varphi} \wedge \varphi.$$

Consequently (4.13) holds true for any $1 \leq r \leq m$.

Now from (4.10), $\psi_r^\pm = k_r^\pm \varphi$ and (2.5), we get

$$(4.16) \quad dk_r^\pm \wedge \varphi - i(r+1)k_r^\pm(\omega_{12} + \omega_r^\pm) \wedge \varphi = 0, \quad 1 \leq r \leq m.$$

We claim that

$$(4.17) \quad dk_r^\pm \wedge \varphi - i(r+1)k_r^\pm \omega_{12} \wedge \varphi \pm ik_r^\pm \omega_{2r+1,2r+2} \wedge \varphi = 0, \quad 1 \leq r \leq m.$$

Actually (4.16) easily implies (4.17) if $\rho_r = 1$ and $1 \leq r \leq m-1$, since by definition $\omega_r^+ = r\omega_{12} - \omega_{2r+1,2r+2}$.

Assume that $\rho_r < 1$ and $1 \leq r \leq m-1$. Then our assumptions (4.4) and (4.5) yield $\Delta \log a_r^+ = (r+1)K$ and since $\Delta \log \lambda = -K$, we get $\Delta \log(a_r^+ \lambda^{r+1}) = 0$. According to Lemma 3 and using the fact that a_r^+ is AVT, we deduce that there exists a holomorphic function g_r^+ such that $a_r^+ \lambda^{r+1} = |g_r^+|$. Moreover, $a_r^- \lambda^{r+1} = |g_r^-|$, where $g_r^- := \sigma_r g_r^+$.

Then we may choose k_r^\pm so that $k_r^+ = \mu^{-r-1} g_r^+$ and $k_r^- = \sigma_r k_r^+$. Since $k_r^+ k_r^- \mu^{2r+2}$ is holomorphic, from (4.11) we get

$$-\frac{1}{2r}(\omega_r^+ + \omega_r^-) = 2\text{Im}(\bar{\partial}(\log \mu) d\bar{z}),$$

or equivalently, in view of $\varphi = \mu dz$,

$$d\varphi = -i \frac{\omega_r^+ + \omega_r^-}{2r} \wedge \varphi.$$

From $d\varphi = -i\omega_{12} \wedge \varphi$ and the uniqueness of the connection form, we deduce that

$$\omega_{12} = \frac{1}{2r}(\omega_r^+ + \omega_r^-)$$

and so

$$\omega_r^\pm = r\omega_{12} \mp \omega_{2r+1,2r+2}.$$

Thus from (4.10) and (2.5), we infer that (4.17) holds for any $1 \leq r \leq m-1$. We note that since $k_r^- = \sigma_r k_r^+$, (4.17) yields $\omega_{2r+1,2r+2} = 0$ for any $1 \leq r \leq m-1$, with $\rho_r < 1$.

Now we want to prove (4.17) for $r = m$. Our assumption (4.6) yields

$$\Delta \log(a_m^+ a_m^-) = 2(m+1)K.$$

Using $\Delta \log \lambda = -K$, we get $\Delta \log(a_m^+ a_m^- \lambda^{2m+2}) = 0$. Since $a_m^+ a_m^- \lambda^{2m+2}$ is AVT, according to Lemma 3, there exists a holomorphic function g_m such that

$$a_m^+ a_m^- \lambda^{2m+2} = |g_m|.$$

We may choose k_m^\pm so that $k_m^+ k_m^- \mu^{2m+2} = g_m$. Taking into account the fact that $k_m^+ k_m^- \mu^{2m+2}$ is holomorphic, from (4.11) we get

$$-\frac{1}{2m}(\omega_m^+ + \omega_m^-) = 2\text{Im}(\bar{\partial}(\log \mu) d\bar{z}).$$

We may now argue as above to deduce that (4.17) holds for $r = m$.

We use (4.17) and (2.5) to compute the exterior derivative of $\omega_{2r+1, 2r+2}$ for any $1 \leq r \leq m$. On the other hand, using the fact $\omega_{2r+1, 2r+2} = 0$ for any $1 \leq r \leq m-1$, with $\rho_r < 1$, and the definition of the 1-forms ω_{AB} , we easily see that (4.14) holds true.

According to the fundamental theorem of submanifolds, there exists an isometric immersion $f_{\theta_1, \dots, \theta_m} : (U, ds^2) \rightarrow S^n$ with corresponding connection forms ω_{AB} .

Clearly f is minimal and the components of its complexified higher fundamental forms are given by

$$H_{2r+1} \omega_{2r+1, \alpha} + H_{2r+2} \omega_{2r+2, \alpha} = H_\alpha \bar{\varphi} \quad \text{for } \alpha = 2r+3, 2r+4,$$

$0 \leq r \leq m-1$, where $H_1 = 1, H_2 = i$, and when $n = 2m+1$

$$H_{2m-1} \omega_{2m-1, 2m+1} + H_{2m} \omega_{2m, 2m+1} = H_{2m+1} \bar{\varphi}.$$

Bearing in mind the definition of the forms ω_{AB} , by an inductive argument, we deduce that the following hold

$$\bar{H}_{2r+1} = \frac{1}{2} e^{i\theta_r} (1 + \sigma_r) k_r^+ \quad \text{and} \quad \bar{H}_{2r+2} = -\frac{i}{2} e^{i\theta_r} (1 - \sigma_r) k_r^-, \quad 1 \leq r \leq m-1,$$

while

$$\bar{H}_{2m+1} = \frac{1}{2} e^{i\theta_m} (k_m^+ + k_m^-) \quad \text{and} \quad \bar{H}_{2m+2} = -\frac{i}{2} e^{i\theta_m} (k_m^+ - k_m^-).$$

Recalling the fact that $k_r^- = \sigma_r k_r^+$ for any $1 \leq r \leq m-1$, from the above we see that the Hopf differentials of f are given by

$$\Phi_r = \frac{1}{4} (\bar{H}_{2r+1}^2 + \bar{H}_{2r+2}^2) \varphi^{2r+2} = \frac{1}{4} e^{i\theta_r} k_r^+ k_r^- \mu^{2r+2} dz^{2r+2}.$$

In particular, they are holomorphic. Thus f is exceptional. If $\rho_r = 1$ for some r , then $\Phi_r = 0$ and by the result in [24] f does not depend on θ_r .

To prove that f is well defined on the whole of M , we cover M with simply connected coordinate neighborhoods U_t . Then we have exceptional surfaces $f_t : U_t \rightarrow S^n$ which can be chosen so that they have the same Hopf differentials in the intersections $U_t \cap U_s$. This is achievable, since M is simply connected. Thus by [24], f_t and f_s are congruent on $U_t \cap U_s$. Continuing in this way, we get a minimal surface $f : M \rightarrow S^n$ which has the desired properties.

(ii) *Uniqueness.* Now let $\tilde{f} : (M, ds^2) \rightarrow S^n$ be another minimal surface arising from the same data $a_r^\pm, 1 \leq r \leq m$. Then $\tilde{f} = f_{\theta_1, \dots, \theta_m}$ and \tilde{f} have congruent higher curvature ellipses. By Theorem 1, \tilde{f} is also exceptional. Moreover the Hopf differentials have the same length and are holomorphic. Hence there exist real numbers η_1, \dots, η_m so that $\tilde{\Phi}_r = e^{i\eta_r} \Phi_r$ for any r . This means that \tilde{f} and

$f_{\theta_1+\eta_1, \dots, \theta_m+\eta_m}$ are congruent, since they have the same Hopf differentials (cf. [24]). \square

5. PSEUDOHOLOMORPHIC CURVES IN S^6

It is well known that the multiplicative structure on the Cayley numbers \mathbb{O} can be used to define an almost complex structure J on the sphere S^6 in \mathbb{R}^7 . This complex structure is not integrable but is nearly Kähler.

A pseudoholomorphic curve in S^6 is a non-constant map $f : M \rightarrow S^6$ whose differential is complex linear, M being a Riemann surface. Pseudoholomorphic curves in S^6 are actually superconformal. According to [3] there are four types of such curves as follows:

- (I) superminimal in S^6
- (II) non-superminimal in S^6
- (III) those that lie in some totally geodesic S^5 of S^6
- (IV) totally geodesic.

As an application of the main results of the previous section, we provide both an intrinsic and an extrinsic characterization for each type of pseudoholomorphic curves, via the a -invariants, among the class of superconformal surfaces. An intrinsic characterization was given by Hashimoto [14], but not for each type separately.

Theorem 4. *Let $f : (M, ds^2) \rightarrow S^6$ be a superconformal surface with Gaussian curvature K . Then the following hold:*

- (i) *f is locally $O(7)$ -congruent to a pseudoholomorphic of type (I) if and only if $\Delta \log(1 - K) = 6K - 1$, or equivalently if and only if f is superminimal with $a_2^+ = a_1^+/2$.*
- (ii) *f is locally $O(7)$ -congruent to a pseudoholomorphic of type (III) if and only if $\Delta \log(1 - K) = 6K$, or equivalently if and only if $a_2^+ = a_2^- = a_1^+/2$ and $a_1^- = 0$.*
- (iii) *f is locally $O(7)$ -congruent to a pseudoholomorphic of type (II) if and only if $6K > \Delta \log(1 - K) > 6K - 1$ and*

$$\Delta \log \left((1 - K)^2 (1 - 6K + \Delta \log(1 - K)) \right) = 12K,$$

or equivalently if and only if $a_1^- = 0$ and either $a_2^+ = a_1^+/2$ or $a_2^- = a_1^+/2$.

It is worth noticing that the condition $\Delta \log(1 - K) = 6K$ is equivalent to the flatness of the metric $(1 - K)^{\frac{1}{3}} ds^2$, while the condition

$$\Delta \log \left((1 - K)^2 (1 - 6K + \Delta \log(1 - K)) \right) = 12K$$

is equivalent to the flatness of the metric

$$\left((1 - K)^2 (1 - 6K + \Delta \log(1 - K)) \right)^{\frac{1}{6}} ds^2.$$

So the pseudoholomorphic curves of type (II) and (III) are characterized by a Ricci-type condition.

The multiplication on the Cayley numbers \mathbb{O} yields a cross product on the purely imaginary Cayley numbers $\text{Im}(\mathbb{O}) = \mathbb{R}^7$ by

$$x \times y = \frac{1}{2}(x \cdot y - y \cdot x).$$

The scalar product on \mathbb{R}^7 is given by

$$\langle x, y \rangle = -\frac{1}{2}(x \cdot y + y \cdot x).$$

The almost complex structure on S^6 is the endomorphism of its tangent bundle given by

$$J_x v = x \times v, \quad x \in S^6, v \in T_x S^6.$$

J is orthogonal and its covariant derivative is given by

$$(5.1) \quad (\nabla_X J)Y = X \times Y + \langle X, JY \rangle x,$$

X, Y being tangent vector fields. Then one can easily obtain the following

Lemma 4. *Let $f : (M, ds^2) \rightarrow S^6$ be a pseudoholomorphic curve and M be a Riemann surface whose complex structure is also denoted by J . For any vector fields X, Y, Z tangent to M we have:*

$$B_1(JX, Y) = B_1(X, JY) = JB_1(X, Y),$$

$$\begin{aligned} \nabla_X^\perp B_1(JY, Z) - J\nabla_X^\perp B_1(Y, Z) &= df(x) \cdot B_1(X, Y) \\ -df \circ J \circ A_{B_1(Y, Z)}X + df(A_{B_1(JY, Z)}X), \end{aligned}$$

where A_ξ is the shape operator associated with a normal direction ξ and ∇^\perp is the normal connection.

Proof. The lemma follows by differentiating twice $df \circ J = J \circ df$, using (5.1), Gauss and Weingarten formulas and the fact that $df(X) \times df(Y) = -\langle X, JY \rangle f$. \square

In particular, Lemma 4 shows that pseudoholomorphic curves in S^6 are superconformal surfaces.

Lemma 5. *For every pseudoholomorphic curve $f : (M, ds^2) \rightarrow S^6$ we have $a_1^- = 0$ and $a_2^+ = a_1^+/2$, or $a_2^- = a_1^+/2$.*

Proof. We choose an orthonormal frame along f so that

$$\begin{aligned} B_1(e_1, e_1) &= |B_1(e_1, e_1)|e_3, \quad B_1(e_1, e_2) = |B_1(e_1, e_2)|e_4, \\ e_6 &= df(e_1) \cdot e_3, \quad e_5 = Je_6. \end{aligned}$$

Then we easily deduce that $k_1^+ = 2\kappa_1, k_1^- = 0, k_2^+ = \kappa_1, k_2^- = 2\overline{H}_5 - \kappa_1$. In particular, we get $H_3 = \kappa_1, H_4 = i\kappa_1$. Moreover, appealing to Lemma 4 and bearing in mind that the third fundamental form is given by

$$B_2(X, Y, Z) = \pi_2 \left(\nabla_X^\perp B_1(Y, Z) \right),$$

where π_2 is the projection onto the second normal bundle, we find that $H_6 = i(\kappa_1 - H_5)$. Then the proof follows by using (2.6)-(2.9). \square

Lemma 6. *Let $f : (M, ds^2) \rightarrow S^6$ be a superconformal surface. If*

$$(5.2) \quad \Delta \log(1 - K) = 6K - 1,$$

then f is superminimal with $a_2^+ = a_1^+/2$. Conversely, if f is superminimal and $a_2^+ = a_1^+/2$, then (5.2) is satisfied.

Proof. At first we assume that (5.2) is satisfied and f is not superminimal. Corollary 1 yields $b_2^2 = 1 - K$ and then equations (4.8) become

$$\Delta \log a_2^\pm = 3K \mp 2 \frac{K_2^\perp}{b_2^2},$$

or equivalently on account of (5.2), $\Delta \log(1 \pm F) = 1 \mp F$, where $F := 4K_2^\perp/b_2^2$. From this we get $|\nabla F|^2 = -(1 - F^2)^2$, which holds only if $F = 1$, or equivalently if f is

superminimal, which is a contradiction. Thus f is superminimal and $a_2^+ = a_1^+/2$ by Lemma 5.

Conversely, if we assume that f is superminimal with $a_2^+ = a_1^+/2$, then Corollary 1 immediately implies (5.2). \square

Lemma 7. *Let $f : (M, ds^2) \rightarrow S^6$ be a superconformal surface. If*

$$(5.3) \quad \Delta \log(1 - K) = 6K,$$

then $a_2^+ = a_2^- = a_1^+/2$, $a_1^- = 0$ and f lies in a totally geodesic S^5 of S^6 . Conversely, if $a_2^+ = a_2^- = a_1^+/2$ and $a_1^- = 0$, then (5.3) is satisfied and f lies in a totally geodesic S^5 .

Proof. We assume that (5.3) is satisfied. Corollary 1 implies that f cannot be superminimal. Then again Corollary 1 yields $b_2^2 = 2(1 - K)$ and equations (4.8) become

$$\Delta \log a_2^\pm = 3K \mp 4 \frac{K_2^\perp}{b_2^2},$$

or equivalently on account of (5.3), $\Delta \log(1 \pm F) = \mp 2F$, where $F := 4K_2^\perp/b_2^2$. We claim that F is constant. Arguing indirectly, we assume that $\nabla F \neq 0$. Then from $\Delta F = -2F(1 + F^2)$, $|\nabla F|^2 = 2F^2(1 - F^2)$ and a well known argument¹ we easily see that that $K = -8$, which contradicts (5.3). Hence F is constant and so $K_2^\perp = 0$. This means that f lies in a totally geodesic S^5 of S^6 with $a_2^+ = a_2^- = a_1^+/2$ and $a_1^- = 0$.

Conversely, if $a_2^+ = a_2^- = a_1^+/2$ and $a_1^- = 0$, then $K_2^\perp = 0$, f lies in a totally geodesic S^5 and Corollary 1 immediately implies (5.3). \square

Lemma 8. *Let $f : (M, ds^2) \rightarrow S^6$ be a superconformal and non-superminimal surface. The condition*

$$(5.4) \quad \Delta \log \left((1 - K)^2 (1 - 6K + \Delta \log(1 - K)) \right) = 12K$$

is satisfied if and only if either $a_2^+ = a_1^+/2$ or $a_2^- = a_1^+/2$.

Proof. Assume that (5.4) is satisfied. From Corollary 1, we get

$$b_2^2 = (1 - K)(2 - 6K + \Delta \log(1 - K)).$$

Then condition (5.4) becomes

$$(5.5) \quad \Delta \log \left(\frac{b_2^2}{1 - K} - 1 \right) = 4 - \frac{2b_2^2}{1 - K}$$

and in view of this, equations (4.8) are written

$$\Delta \log u_2^\pm = 2 - u_2^\pm,$$

where $u_2^\pm := 4(a_2^\pm)^2/(1 - K)$, or equivalently

$$(5.6) \quad \Delta u_2^\pm = \frac{|\nabla u_2^\pm|^2}{u_2^\pm} + u_2^\pm(2 - u_2^\pm).$$

¹It is known (cf. [11]) that if a two-dimensional Riemannian manifold M allows a smooth function $f : M \rightarrow \mathbb{R}$ such that $\Delta f = P(f)$ and $|\nabla f|^2 = Q(f)$ for smooth functions $P, Q : \mathbb{R} \rightarrow \mathbb{R}$, then on the set of points where the gradient ∇f doesn't vanish, the Gaussian curvature K satisfies

$$2KQ + (2P - Q')(P - Q'') + Q(2P' - Q'') = 0.$$

Since $u_2^+ + u_2^- = b_2^2/(1-K)$, (5.5) is written

$$\Delta \log(u_2^+ + u_2^- - 2) = 4 - (u_2^+ + u_2^-),$$

or equivalently by virtue of (5.6),

$$\begin{aligned} u_2^-(u_2^- - 2)|\nabla u_2^+|^2 - 2u_2^+u_2^-\langle \nabla u_2^+, \nabla u_2^- \rangle + u_2^+(u_2^+ - 2)|\nabla u_2^-|^2 \\ = 2u_2^+u_2^-(u_2^+ + u_2^- - 2)(u_2^- - 2)(2 - u_2^+). \end{aligned}$$

By the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} u_2^-(u_2^- - 2)|\nabla u_2^+|^2 - 2u_2^+u_2^-\langle \nabla u_2^+, \nabla u_2^- \rangle + u_2^+(u_2^+ - 2)|\nabla u_2^-|^2 \\ \leq 2u_2^+u_2^-(u_2^+ + u_2^- - 2)(u_2^- - 2)(2 - u_2^+) \\ \leq u_2^-(u_2^- - 2)|\nabla u_2^+|^2 + 2u_2^+u_2^-\langle \nabla u_2^+, \nabla u_2^- \rangle + u_2^+(u_2^+ - 2)|\nabla u_2^-|^2. \end{aligned}$$

It is easy to see that this double inequality holds only if $u_2^+ = 2$ or $u_2^- = 2$. This yields $a_2^+ = a_1^+/2$ or $a_2^- = a_1^-/2$.

Conversely, we assume that $a_2^+ = a_1^+/2$ (the case $a_2^- = a_1^-/2$ is similar). Corollary 1 immediately implies $b_2^2 = (1-K)(2-6K+\Delta \log(1-K))$. From

$$a_2^\pm := \sqrt{\frac{1}{4}b_2^2 \pm K_2^\perp}$$

and $a_2^+ = a_1^+/2$ we find

$$K_2^\perp = \frac{1-K}{4}(6K - \Delta \log(1-K))$$

and

$$a_2^- = \frac{1}{\sqrt{2}}\sqrt{(1-K)(1-6K+\Delta \log(1-K))}.$$

On account of Corollary 1, we get $\Delta \log(a_2^+a_2^-) = 6K$, which yields (5.4). \square

Proof of Theorem 4. At first we assume that $f : (M, ds^2) \rightarrow S^6$ is a pseudoholomorphic curve. According to Lemma 5, we have $a_1^- = 0$, and either $a_2^+ = a_1^+/2$ or $a_2^- = a_1^-/2$.

If f is of type (I), then $a_2^- = 0$, and then Lemma 6 yields $\Delta \log(1-K) = 6K-1$.

If f is of type (III), then $K_2^\perp = 0$ and so $a_2^+ = a_2^- = a_1^+/2$. Then Lemma 7 immediately implies $\Delta \log(1-K) = 6K$.

Now assume that f is of type (II). From Corollary 1, we get

$$\|B_2\|^2 = (1-K)(2-6K+\Delta \log(1-K)).$$

Assume further that $a_2^+ = a_1^+/2$. From the proof of Lemma 8, we have $6K > \Delta \log(1-K)$ and

$$(a_2^+a_2^-)^2 = (1-K)^2(1-6K+\Delta \log(1-K)).$$

Corollary 1 implies that

$$\Delta \log \left((1-K)^2(1-6K+\Delta \log(1-K)) \right) = 12K.$$

Now assume that f is superconformal and satisfies one of (5.2), (5.3) or (5.4). In the case where (5.4) is fulfilled, we further assume that $6K > \Delta \log(1-K) > 6K-1$. Then the proof of the theorem follows from the preceding lemmas and [14]. Indeed, our conditions imply that (M, ds^2) satisfies the condition in [14, Th. 6.1]. Hence there exists locally a pseudoholomorphic curve g in S^6 with induced metric ds^2 .

If (5.2) is satisfied or equivalently if f is superminimal with $a_2^+ = a_1^+/2$, then g is also superminimal, by Lemma 6. Since superminimal surfaces are rigid, we see that f is $O(7)$ -congruent to g .

If (5.3) is satisfied or equivalently if $a_2^+ = a_2^- = a_1^+/2$ and $a_1^- = 0$, then by Lemma 7, f, g lie in a totally geodesic S^5 and have the same a -invariants. From Corollary 1, we see that f is $O(7)$ -congruent to some g_θ .

Now assume that (5.4) is satisfied or equivalently $a_1^- = 0$ and either $a_2^+ = a_1^+/2$ or $a_2^- = a_1^+/2$. Lemma 8 then shows that f and g have the same a -invariants, and the argument is the same as above. \square

6. THE RICCI CONDITION

As an application of the main result in this paper, we provide another proof of the following result [25] that supports the Lawson's conjecture [19].

Theorem 5. *Lawson's conjecture is true for non-flat exceptional surfaces lying in odd-dimensional spheres.*

We recall that a two-dimensional Riemannian manifold (M, ds^2) with Gaussian curvature $K \leq 1$ satisfies the Ricci condition, if and only the metric $d\tilde{s}^2 = \sqrt{1-K}ds^2$ is flat away from points where $K = 1$, or equivalently $\Delta \log(1-K) = 4K$.

Lemma 9. *Let $f : (M, ds^2) \rightarrow S^n$ be a non-flat exceptional surface which satisfies the Ricci condition. Then the a -invariants of f are given by*

$$a_r^+ = \begin{cases} c_r(1-K)^{\frac{r+1}{4}} & \text{if } r \text{ is odd,} \\ c_r(1-K)^{\frac{r+2}{4}} & \text{if } r \equiv 2 \pmod{4}, \\ c_r(1-K)^{\frac{r}{4}} & \text{if } r \equiv 0 \pmod{4}, \end{cases}$$

$$a_r^- = a_r^+ \sqrt{\frac{1-\rho_r}{1+\rho_r}},$$

where $c_r = 2^{\frac{2-r}{2}}\beta_r$, for $0 \leq r \leq m-1$, $c_m = 2^{\frac{1-m}{2}}\beta_m$, $\beta_1 = \sqrt{2}$, $\beta_r = \rho_{r-1}\beta_{r-1}$ and $\rho_r = 1$ if r is even, $m = [(n-1)/2]$.

Proof. The lemma follows easily by induction using the Ricci condition and Theorem 2. \square

Proof of Theorem 5. Let $f : (M, ds^2) \rightarrow S^n$ be a non-flat exceptional minimal surface which satisfies the Ricci condition, where n is odd. We claim that $n \equiv 3 \pmod{4}$. Arguing indirectly, we suppose that $n = 4m+1$. Then Lemma 9 yields $a_{2m} = c_{2m}(1-K)^{\frac{m}{2}}$ if m is even and $a_{2m} = c_{2m}(1-K)^{\frac{m+1}{2}}$ if m is odd. Moreover, viewing f as a minimal surface in S^{4m+2} , we obviously have $K_{2m}^\perp = 0$. Then from Theorem 2, we obtain $\Delta \log a_{2m} = (2m+1)K$. However, this combined with the Ricci condition yields $K = 0$, which is a contradiction.

Hence $n = 4m+3$. According to Lemma 9, $\Phi_r = 0$ if r is even. Let $r_0 = \min\{r : 1 \leq r \leq 2m+1 \text{ with } \Phi_r \neq 0\}$. Obviously r_0 is odd. Let z be a local complex coordinate such that $ds^2 = F|dz|^2$. From the definition of Hopf differentials we know that $\Phi_r = f_r dz^{2r+2}$, where $f_r = \langle B_r^{(r+1,0)}, B_r^{(r+1,0)} \rangle$. We pick a branch g of $f_{r_0}^{\frac{2}{r_0+1}}$ and define the quadratic form $\Phi = g dz^4$. It is obvious that Φ is well defined

and holomorphic. For any odd $r \geq r_0$, we write $\Phi_r = |f_r|e^{i\tau_r}dz^{2r+2}$. Appealing to Lemma 9 and (2.10), we obtain

$$\Phi_r = \gamma_r e^{i\left(\tau_r - \frac{r+1}{r_0+1}\tau_{r_0}\right)} \Phi^{\frac{r+1}{2}},$$

where γ_r is a positive number. From the holomorphicity of Φ_r and Φ , we deduce that $\tau_r - \frac{r+1}{r_0+1}\tau_{r_0}$ is constant. Moreover, we easily see that $|g|^2 = c_0 F^4(1-K)^2$, where c_0 is a positive constant. Using the holomorphicity of g and arguing as in [18, Theorem 8], we infer that there exists locally a minimal surface \tilde{f} in S^3 with Hopf differential $\tilde{\Phi} = c\Phi$, where c is a complex number. Therefore $\Phi_r = \delta_r \tilde{\Phi}^{\frac{r+1}{2}}$ for any odd $r \geq r_0$, where δ_r is a complex number and $\Phi_r = 0$ otherwise. Thus in view of Proposition 2 in [25], the Hopf differentials of f coincide with those of minimal surfaces which decompose as a direct sum of the associated minimal surfaces in S^3 . Appealing to the main result in [24], we see that f splits as a direct sum of the associated minimal surfaces of \tilde{f} . \square

7. POLAR AND SELF-DUAL SURFACES

It is well known [18] that the Gauss map of a minimal surface M in S^3 defines another minimal surface, the polar of M , which is conformal to M . The polar can also be defined for any minimal surface lying in odd-dimensional spheres, just by selecting a unit section of the last normal bundle. More precisely, the *polar* of a minimal surface $f : (M, ds^2) \rightarrow S^{2m+1}$ is the map $f^* : M^* \rightarrow S^{2m+1}$ defined by $f^* = e_{2m+1}$, where M^* is the set of generic points, and e_{2m+1} is a unit section of the last normal bundle. We note that f^* is defined all over M if f is exceptional, since all higher normal bundles are well defined over singular points [25, Prop. 4].

Miyaoka [20] proved that the polar of any superconformal surface is again a superconformal surface. We give an alternative proof of this fact. Furthermore, as in the case of minimal surfaces in S^3 , the polar is conformal to the given surface and they have the same Hopf differentials.

Proposition 2. *Let $f : (M, ds^2) \rightarrow S^{2m+1}$ be a superconformal surface. Then its polar f^* is a superconformal surface, with induced metric $ds_*^2 = (2a_m^+/a_{m-1}^+)^2 ds^2$ and the same Hopf differentials as f . Furthermore, the polar of f^* is f .*

Proof. The proof follows easily, if we use the results by Dajczer and Florit [10]. Indeed, according to [10, Prop. 8], the polar is an elliptic surface whose higher normal bundles are given by $N^r f^* = N^{m-1-r} f$ for any $0 \leq r \leq m-1$, where $N^0 f := df(TM^2)$ and $N^0 f^* := df^*(TM^2)$. Furthermore, $N^m f^* = \text{span}\{f^*\}$. Since f is superconformal, all its higher curvature ellipses are circles up to the last but one. Thus the corresponding complex structures J_r and $\tilde{J}_r = J_{m-1-r}^t$ defined in [10] are orthogonal. This means that f^* is minimal, all its higher curvature ellipses are circles up to the last but one, and so it is superconformal.

To compute the last Hopf differential Φ_m^* we choose a local conformal coordinate z and proceed as follows:

$$\begin{aligned}
\Phi_m^* &= \langle B_m^{*(m+1,0)}, B_m^{*(m+1,0)} \rangle dz^{2m+2} = \langle \bar{\nabla}_{\partial} \dots \bar{\nabla}_{\partial} df^*(\partial), f \rangle^2 dz^{2m+2} \\
&= \left(\partial \langle \bar{\nabla}_{\partial} \dots \bar{\nabla}_{\partial} df^*(\partial), f \rangle - \langle \bar{\nabla}_{\partial} \dots \bar{\nabla}_{\partial} df^*(\partial), df(\partial) \rangle \right)^2 dz^{2m+2} \\
&= \langle \bar{\nabla}_{\partial} \dots \bar{\nabla}_{\partial} df^*(\partial), df(\partial) \rangle^2 dz^{2m+2} = \dots \\
&= \langle df^*(\partial), \bar{\nabla}_{\partial} \dots \bar{\nabla}_{\partial} df(\partial) \rangle^2 dz^{2m+2} \\
&= \langle f^*(\partial), \bar{\nabla}_{\partial} \dots \bar{\nabla}_{\partial} df(\partial) \rangle^2 dz^{2m+2} \\
&= \langle e_{2m+1}, \bar{\nabla}_{\partial} \dots \bar{\nabla}_{\partial} df(\partial) \rangle^2 dz^{2m+2} \\
&= \langle B_m^{(m+1,0)}, B_m^{(m+1,0)} \rangle dz^{2m+2} = \Phi_m.
\end{aligned}$$

Since f is superconformal, we may choose the frame so that $H_{2r+1} = \kappa_r$ and $H_{2r+2} = i\kappa_r$ for any $1 \leq r \leq m-1$. Then (2.11) and (2.12) yield

$$\begin{aligned}
\omega_{2m-1,2m+1} &= \frac{1}{2\kappa_{m-1}} (H_{2m+1}\bar{\varphi} + \bar{H}_{2m+1}\varphi), \\
\omega_{2m,2m+1} &= \frac{i}{2\kappa_{m-1}} (H_{2m+1}\varphi - H_{2m+1}\bar{\varphi}).
\end{aligned}$$

Bearing in mind (2.3) and the above, for any X, Y tangent to M^2 , we find

$$\langle df^*(X), df^*(X) \rangle = \frac{\kappa_m^2}{\kappa_{m-1}^2} \langle X, Y \rangle = \left(\frac{2a_m^+}{a_{m-1}^+} \right)^2 \langle X, Y \rangle.$$

□

The polar of a superconformal surface $f : (M, ds^2) \rightarrow S^{2m+1}$ can be characterized, up to congruence, as the minimal surface $f^* : M \rightarrow S^{2m+1}$ with the same Hopf differentials as f and induced metric $(2a_m^+/a_{m-1}^+)^2 ds^2$.

The following proposition [20] is a consequence of Proposition 2 and the fact that two minimal surfaces with the same induced metric and the same Hopf differentials are congruent [24].

Proposition 3. *Let $f : (M, ds^2) \rightarrow S^{2m+1}$ be a superconformal surface. Then f and its polar f^* are isometric if and only if they are congruent.*

We are interested in superconformal surfaces in odd-dimensional spheres with the property that they are isometric to their polar. We call these surfaces *self-dual*.

The following corollary provides a link between the Ricci condition and furnishes examples of self-dual surfaces.

Corollary 2. (i) *A non-flat superconformal surface in S^{2m+1} that satisfies the Ricci condition is self-dual if and only if $m = 4k + 3$.*

(ii) *Superconformal surfaces in S^{4k+1} are self-dual if they are flat.*

Proof. Part (i) follows from Lemma 9 and Proposition 2, while part (ii) follows immediately from Corollary 1 and Proposition 2. □

We now give a complete characterization of self-dual surfaces. It turns out that this property is intrinsic. Indeed, we characterize all metrics that arise as induced metrics of self-dual surfaces.

Theorem 6. *Let $f : (M, ds^2) \rightarrow S^{2m+1}$ be a superconformal surface. If f is self-dual, then its a -invariants satisfy*

$$(7.1) \quad \frac{a_{m-r}^+}{a_{m-r-1}^+} = \frac{a_r^+}{a_{r-1}^+}, \quad 0 \leq r \leq m,$$

where $a_0^+ =: 2, a_{-1}^+ := 4$. Moreover,

(i) if $m = 2l$, then $a_m^+ = \frac{1}{4}a_{l-1}^+a_l^+$ and $\Delta \log(a_{l-1}^+a_l^+) = (m+1)K$, while

(ii) if $m = 2l+1$, then $a_m^+ = \frac{1}{4}(a_l^+)^2$ and $\Delta \log a_l^+ = (m+1)K/2$.

Conversely, if $a_m^+ = \frac{1}{4}a_{l-1}^+a_l^+$ when $m = 2l$, or $a_m^+ = \frac{1}{4}(a_l^+)^2$ when $m = 2l+1$, then f is self-dual.

Proof. Assume that f is self-dual. Then Proposition 2 shows that (7.1) holds for $r = 0$. From this we get $\Delta \log a_m^+ = \Delta \log a_{m-1}^+$, and appealing to Corollary 1, we see that (7.1) holds for $r = 1$. Arguing in this way by reduction and by virtue of Corollary 1, we prove (7.1) for any $0 \leq r \leq m$. If $m = 2l$, then (7.1) yields $a_m^+ = \frac{1}{4}a_{l-1}^+a_l^+$ and consequently $\Delta \log(a_{l-1}^+a_l^+) = (m+1)K$ follows from Corollary 1. The case $m = 2l+1$ is similar.

Conversely, we assume that $a_m^+ = \frac{1}{4}a_{l-1}^+a_l^+$ and $m = 2l$. The other case is treated in a similar manner. Using (4.7) and (4.8) in Corollary 1, we get

$$\frac{a_{l+1}^+}{a_l^+} = \frac{a_{l-1}^+}{a_{l-2}^+}.$$

From this and again appealing to Corollary 1, we obtain

$$\frac{a_{l+2}^+}{a_{l+1}^+} = \frac{a_l^+}{a_{l-1}^+}.$$

Inductively, we deduce that (7.1) holds for any $0 \leq r \leq m$. In particular, this yields $2a_m^+ = a_{m-1}^+$, and so by Proposition 2, f is self-dual. \square

Now we characterize all metrics which arise as induced metrics on self-dual surfaces.

Theorem 7. *Let (M, ds^2) be a simply connected two-dimensional Riemannian manifold with Gaussian curvature $K \leq 1$. We consider the non-negative functions $a_r^+, 0 \leq r \leq l$, defined inductively by*

$$(7.2) \quad \Delta \log a_r^+ = (r+1)K - 2\frac{a_r^2}{a_{r-1}^2} + 2\frac{a_{r+1}^2}{a_r^2}, \quad 0 \leq r \leq l-1,$$

where $a_0^+ =: 2, a_{-1}^+ := 4, a_1^+ := \sqrt{2(1-K)}$ and l being a positive integer. Assume that these functions are AVT and either $\Delta \log(a_{l-1}^+a_l^+) = (2l+1)K$ or $\Delta \log a_l^+ = (l+1)K$. Then for any $\theta \in \mathbb{R}$ there exists a self-dual surface $f_\theta : (M, ds^2) \rightarrow S^n$, with $n = 4l+1$ or $n = 4l+3$, whose a -invariants up to order l are precisely the AVT functions a_r^+ and $a_r^- = 0, 1 \leq r \leq l$. Furthermore, any other self-dual immersion of (M, ds^2) arising from these data is congruent to some f_θ .

Proof. Assume that $\Delta \log(a_{l-1}^+a_l^+) = (2l+1)K$. Then we define inductively the functions $a_r^+, l+1 \leq r \leq 2l$, by

$$(7.3) \quad \frac{a_{2l-r}^+}{a_{2l-r-1}^+} = \frac{a_r^+}{a_{r-1}^+}.$$

Obviously these functions are also AVT. Using (7.2) and $\Delta \log(a_{l-1}^+ a_l^+) = (2l+1)K$, we can prove by induction that $a_r^+, 1 \leq r \leq 2l$, satisfy (4.7) and (4.8). Then according to Corollary 1, for any $\theta \in \mathbb{R}$ there exists a superconformal surface $f_\theta : (M, ds^2) \rightarrow S^n$, with $n = 4l + 1$. Obviously f is self-dual. Furthermore, any other self-dual immersion of (M, ds^2) arising from these data is congruent to some f_θ .

The case where $\Delta \log a_l^+ = (l+1)K$ is treated in a similar manner. In this case, we end up with a superconformal self-dual surface $f_\theta : (M, ds^2) \rightarrow S^n$ with $n = 4l + 3$. \square

In particular, the following shows that self-dual surfaces in S^5 are up to a congruence the pseudoholomorphic curves of type (III).

Corollary 3. *For any superconformal surface $f : (M, ds^2) \rightarrow S^5$ the following are equivalent:*

- (i) f is self-dual.
- (ii) f and f^* have the same Gaussian curvature at corresponding points.
- (iii) f is locally congruent to pseudoholomorphic curve of type (III).

Proof. Since by Proposition 2 the induced metric of f^* is $ds_*^2 = (2a_2^+/a_1^+)^2 ds^2$, we easily verify that the Gaussian curvature of f^* is given by

$$K_* = \left(\frac{a_1^+}{2a_2^+} \right)^2 \left(K - \Delta \log a_2^+ + \Delta \log a_1^+ \right).$$

Appealing to Corollary 1, we finally get

$$K_* = 1 - \frac{(a_1^+)^4}{8(a_2^+)^2}.$$

From $K = 1 - (a_1^+)^2/2$ and the above, we deduce that $K_* = K$ is equivalent to $2a_2^+ = a_1^+$, which by virtue of Proposition 2 shows the equivalence between (i) and (ii). The equivalence between (i) and (iii) follows from Theorem 4(ii). \square

8. GLOBAL FORMULAS

In this section, we give some topological restrictions for exceptional surfaces. The zero set of an AVT function a on a connected compact oriented surface M is either isolated or the whole of M , and outside its zeros, the function is smooth.

If a is a non-zero AVT function, i.e., locally $a = |t_0|a_1$, with t_0 holomorphic, the order $k \geq 1$ of any $p \in M$ with $a(p) = 0$ is the order of t_0 at p . Let $N(a)$ be the sum of all orders for all zeros of a . Then $\Delta \log a$ is bounded on $M \setminus \{a = 0\}$ and its integral is computed in the following lemma which was proved in [11, 13].

Lemma 10. *Let (M, ds^2) be a compact oriented two-dimensional Riemannian manifold with area element dA . If a is an AVT function on M , then*

$$\int_M \Delta \log a dA = -2\pi N(a).$$

For exceptional surfaces it has been proved in [25, Prop. 4] that all higher normal bundles can be smoothly extended over the whole surface. Then the following follows from Lemma 10, Theorem 2 and Proposition 1.

Corollary 4. *Let $f : (M, ds^2) \rightarrow S^n$ be an exceptional surface. The Euler number $\chi(N^r f)$ of the r -th normal bundle and the Euler-Poincaré characteristic $\chi(M)$ of M satisfy the following:*

(i) *If $\Phi_r \neq 0$ for some $1 \leq r < m$, then*

$$\chi(N^r f) = 0 \text{ and } (r+1)\chi(M) = -N(a_r^+) = -N(a_r^-).$$

(ii) *If $\Phi_r = 0$, for some $1 \leq r \leq m$, then*

$$(r+1)\chi(M) - \chi(N^r f) = -N(a_r^+).$$

(iii) *If $\Phi_m \neq 0$, then*

$$(m+1)\chi(M) \mp \chi(N^m f) = -N(a_m^\pm).$$

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